MULTIDIMENSIONAL STICKY BROWNIAN MOTIONS AS LIMITS OF EXCLUSION PROCESSES

MIKLÓS Z. RÁCZ AND MYKHAYLO SHKOLNIKOV

ABSTRACT. We study exclusion processes on the integer lattice in which particles change their velocities due to stickiness. Specifically, whenever two or more particles occupy adjacent sites, they stick together for an extended period of time and the entire particle system is slowed down until the "collision" is resolved. We show that under diffusive scaling of space and time such processes converge to what one might refer to as a sticky reflected Brownian motion in the wedge. The latter behaves as a Brownian motion with constant drift vector and diffusion matrix in the interior of the wedge, and reflects at the boundary of the wedge after spending an instant of time there. In particular, this leads to a natural multidimensional generalization of sticky Brownian motion on the half-line, which is of interest in both queueing theory and stochastic portfolio theory. For instance, this can model a market, which experiences a slowdown due to a major event (such as a court trial between some of the largest firms in the market) deciding about the new market leader.

1. Introduction

Stochastic processes with sticky points in the Markov process sense have been studied over the course of the last three decades. In the now classical papers [8] and [1], the authors analyze sticky Brownian motion on the half-line, which is the process evolving as a standard Brownian motion away from zero and reflecting at zero after spending an instant of time there (as opposed to a reflecting Brownian motion, which reflects instantaneously at zero). These papers show that sticky Brownian motion arises as a time change of a reflecting Brownian motion, and that it describes the scaling limit of random walks on the natural numbers whose jump rate at zero is significantly smaller than the jump rates at positive sites.

In stochastic analysis the stochastic differential equation (SDE)

$$\mathrm{d}S(t) = \mathbf{1}_{\{S(t)>0\}}\,\mathrm{d}B(t) + \eta\,\mathbf{1}_{\{S(t)=0\}}\,\mathrm{d}t \tag{1.1}$$

satisfied by sticky Brownian motion has drawn much attention, as it is an example of a SDE for which weak existence and uniqueness hold, but strong existence and pathwise uniqueness fail (see [5]). In fact, in [18] (see also the survey [6]) it is shown that a weak solution to (1.1) cannot be adapted to a cozy filtration, that is, a filtration generated by a finite or infinite-dimensional Brownian motion.

Date: February 11, 2013.

Key words and phrases. Exclusion processes with speed change, reflected Brownian motion, scaling limits, sticky Brownian motion, stochastic differential equations, stochastic portfolio theory.

Research supported in part by NSF grant DMS 1106999 and DOD ONR grant N000141110140 (M.Z. Rácz).

The present study is motivated by the question of how one can define and analyze multidimensional analogues of (1.1) and whether solutions to the corresponding systems of SDEs arise as suitable scaling limits of interacting particle systems in analogy to the findings of [8] in the one-dimensional case. In [11, Section 3] it is shown that a large class of reflecting Brownian motions in the n-dimensional wedge

$$\mathcal{W} = \{x \in \mathbb{R}^n : \; x_1 \leq x_2 \leq \ldots \leq x_n\}$$

arise as limits of certain exclusion processes with speed change under diffusive rescaling. As we show below, sticky Brownian motions with state space $\mathcal{W}$ can also be obtained as scaling limits of suitable exclusion processes with speed change.

1.1. **Exclusion processes with sticky particles.** To simplify the exposition, we next describe a simple class of particle systems which converge to sticky Brownian motions in $\mathcal{W}$ in the scaling limit, and postpone the description of the much wider class of particle systems that we can handle to section 3. We fix the number of particles $n \in \mathbb{N}$, and also rate parameters $a > 0$, $\Theta^L = \left(\theta^L_{i,j}\right)_{i \in [n],\, j \in [n-1]} \in [0,\infty)^{n \times (n-1)}$, and $\Theta^R = \left(\theta^R_{i,j}\right)_{i \in [n],\, j \in [n-1]} \in [0,\infty)^{n \times (n-1)}$ with the notation

$$[n] = \{1, 2, \ldots, n\}.$$

For a fixed value of the scaling parameter $M > 0$, the particles move on the rescaled lattice $\mathbb{Z}/\sqrt{M}$; to describe their motion we introduce the following Poisson processes, all of which are independent, and all of which have jump size $\frac{1}{\sqrt{M}}$. For $i \in [n]$, the Poisson processes P_i and Q_i have jump rates Ma, while for $i \in [n]$, $j \in [n-1]$, the Poisson processes $L_{i,j}$ and $R_{i,j}$ have jump rates $\sqrt{M}\theta^L_{i,j}$ and $\sqrt{M}\theta^R_{i,j}$, respectively. In addition, for notational convenience we introduce ghost particles at $\pm\infty$, namely: $X^M_0(\cdot) \equiv -\infty$ and $X^M_{n+1}(\cdot) \equiv \infty$. For any initial condition $X^M_1(0) < X^M_2(0) < \ldots < X^M_n(0)$ on $\mathbb{Z}/\sqrt{M}$ we can then define a particle system evolving on $\mathbb{Z}/\sqrt{M}$ in continuous time by setting

$$\begin{aligned} \mathrm{d}X^M_i(t) &= \mathbf{1}_{\left\{X^M_k(t) + \frac{1}{\sqrt{M}} < X^M_{k+1}(t),\, k \in [n-1]\right\}} \mathrm{d}\left(P_i(t) - Q_i(t)\right) \\ &\quad + \sum_{j=1}^{n-1} \mathbf{1}_{\left\{X^M_i(t) + \frac{1}{\sqrt{M}} < X^M_{i+1}(t),\, X^M_j(t) + \frac{1}{\sqrt{M}} = X^M_{j+1}(t)\right\}} \mathrm{d}R_{i,j}(t) \\ &\quad - \sum_{j=1}^{n-1} \mathbf{1}_{\left\{X^M_{i-1}(t) + \frac{1}{\sqrt{M}} < X^M_i(t),\, X^M_j(t) + \frac{1}{\sqrt{M}} = X^M_{j+1}(t)\right\}} \mathrm{d}L_{i,j}(t), \end{aligned} \tag{1.2}$$

for $i \in [n]$. Note that (1.2) guarantees that for any $t \geq 0$, the particle configuration $(X^M_1(t), X^M_2(t), \ldots, X^M_n(t))$ is an element of the discrete wedge

$$\mathcal{W}^M = \left\{x \in \left(\mathbb{Z}/\sqrt{M}\right)^n : \; x_k + \frac{1}{\sqrt{M}} \leq x_{k+1},\, k \in [n-1]\right\}.$$

Intuitively, when apart, the particles move independently on the rescaled lattice $\mathbb{Z}/\sqrt{M}$ according to the processes $P_i - Q_i$, $i = 1, 2, \ldots, n$ (in particular, with jump rates of order M); however, when two particles land on adjacent sites—an event we describe as a "collision"—the system experiences a *slowdown*: the particles change their jump rates to the ones of the processes $L_{i,j}$ and $R_{i,j}$, $i \in [n]$, $j \in [n-1]$, which

are of order $\sqrt{M}$. The interaction between adjacent particles can be described as *stickiness*, as it takes a long time (on the time scale Mt) until the collision is resolved and the particles return to jump rates of order M.

1.2. **Convergence to multidimensional sticky Brownian motions.** The described particle systems converge to a sticky Brownian motion in $\mathcal{W}$ under the following assumption.

Assumption 1. Define the *speed change matrix* $V = (v_{i,j})_{i\in[n],\,j\in[n-1]}$ by setting $v_{i,j} = \theta^R_{i,j} - \theta^L_{i,j}$, and the *reflection matrix* $Q = (q_{j,j'})_{j,j'\in[n-1]}$ by setting $q_{j,j'} = v_{j+1,j'} - v_{j,j'}$. When there is a collision between particles j' and $j'+1$ and no other collisions, then the velocity of particle i is given by $v_{i,j'}$, and the velocity of gap j between particles j and $j+1$ is given by $q_{j,j'}$. Assume that the matrix Q is *completely-$\mathcal{S}$*, in the sense that there is a $\lambda \in [0,\infty)^{n-1}$ such that $Q\,\lambda \in (0,\infty)^{n-1}$ and the same property is shared by every principal submatrix of Q (see [17] for several equivalent definitions).

Under Assumption 1, we have the following convergence result.

Theorem 1. *Suppose that Assumption* 1 *holds, and also that the initial conditions* $\left\{\left(X_1^M(0), X_2^M(0), \ldots, X_n^M(0)\right), M > 0\right\}$ *are deterministic and converge to a limit* $(x_1, x_2, \ldots, x_n) \in \mathcal{W}$ *as* $M \to \infty$. *Then the laws of the paths of the particle systems* $\left\{\left(X_1^M(\cdot), X_2^M(\cdot), \ldots, X_n^M(\cdot)\right), M > 0\right\}$ *on* $D([0,\infty),\mathbb{R}^n)$ *(the space of càdlàg paths with values in* $\mathbb{R}^n$ *endowed with the topology of uniform convergence on compact sets) converge to the law of the unique weak solution of the system of SDEs*

$$\mathrm{d}X_i(t) = \mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\sqrt{2a}\,\mathrm{d}W_i(t) + \sum_{j=1}^{n-1}\mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}}\,v_{i,j}\,\mathrm{d}t, \tag{1.3}$$

$i \in [n]$, *in* $\mathcal{W}$ *starting from* $(x_1, x_2, \ldots, x_n)$. *Here* $(W_1, W_2, \ldots, W_n)$ *is a standard Brownian motion in* $\mathbb{R}^n$.

The solution to (1.3) *evolves as a Brownian motion when away from the boundary* $\partial\mathcal{W}$ *of* $\mathcal{W}$, *does not spend a non-empty time interval on* $\partial\mathcal{W}$, *however satisfies*

$$\mathbb{P}\left(\mathcal{L}\left(\{t \geq 0:\ X(t) \in \partial\mathcal{W}\}\right) > 0\right) > 0,$$

where $\mathcal{L}$ *is the Lebesgue measure on* $[0,\infty)$.

We refer to the solution of (1.3) with $a = 1/2$ as *sticky Brownian motion* in $\mathcal{W}$ with *reflection matrix* V. We choose this terminology because the SDE (1.3) generalizes one-dimensional sticky Brownian motion as in [8, 1], and also because it is consistent with the terminology used in [17] and the references therein dealing with instantaneously reflecting Brownian motions.

In section 3 we prove a much stronger result than Theorem 1, allowing for non-exponential interarrival times between the jumps in the processes P_i, Q_i, $L_{i,j}$ and $R_{i,j}$ as well as for dependence between the latter processes (see Theorem 6). This then leads to the definition of a sticky Brownian motion in $\mathcal{W}$ whose components have unequal drift and diffusion coefficients. In addition, it is not hard to see from the proof that for each jump parameter $\theta^L_{i,j}$ or $\theta^R_{i,j}$ which is zero, we can choose the jump

rate of the corresponding process $L_{i,j}$ or $R_{i,j}$ to be of order $o\big(\sqrt{M}\big)$ (not necessarily identically zero) for the result of Theorem 1 to still hold.

One of the main technical difficulties in the proof of Theorem 1 and its extension (Theorem 6 in section 3) is posed by the indicator function appearing in the diffusion matrix of the limiting process. This is in contrast to the main convergence result in [11], where the martingale part of the limiting process is a Brownian motion. Another major difference to the setting in [11] is that we consider the full class of completely-$\mathcal{S}$ reflection matrices and are dealing with weak solutions of the limiting stochastic differential equation; whereas in [11] only a special class of reflection matrices is considered, allowing for a pathwise construction of the limiting object. Finally, we allow for dependence of interarrival times between jumps for different particles in Theorem 6 below, which has not been addressed in [11].

1.3. **Applications.** We mention two potential areas of applications for the process in (1.3) and its extensions that appear in section 2. To this end, we recall that reflected Brownian motions in $\mathcal{W}$ give a class of tractable descriptive models for the logarithmic market capitalizations (that is, the logarithms of the total values of stocks) of firms in a large equity market (see, e.g., [11]). These models lead to realistic capital distribution curves in the long-run and are also able to produce a realistic pattern of collisions. In the same spirit, one can think of (1.3) as a model for the logarithmic market capitalizations in an equity market in which the market experiences a slowdown whenever there is a possibility that two firms will exchange their ranks (described by a collision). For example, one can imagine a court trial between two firms, the result of which decides which firm becomes the market leader, leading to a slowdown of the market right before the time of the verdict as the market participants await the result of the trial.

Another area of potential applications is the study of diffusion approximations of storage and queueing networks. It is well-known (see, e.g., the survey [19] and the references therein) that reflected Brownian motions in the orthant describe the heavy traffic limits of many queueing networks such as open queueing networks, single class networks and feedforward multiclass networks. Moreover, in [8] the authors explain how sticky Brownian motion on the half-line can be obtained as the diffusion limit of modified storage processes, and in [13] a related single server queueing system is studied. In view of these results, we expect *sticky Brownian motion in the orthant* $\mathbb{R}_+^{n-1}$, given by the spacings process

$$(X_2(\cdot) - X_1(\cdot), X_3(\cdot) - X_2(\cdot), \dots, X_n(\cdot) - X_{n-1}(\cdot)),$$

to arise in the diffusion limit of suitable queueing networks as well.

1.4. **Future directions.** A natural direction for future work is to study other types of sticky interaction between particles. Even in the class of exclusion processes in one dimension, there are avenues to be explored. For instance, the exclusion processes described by (1.2) experience a *global* slowdown when a collision occurs, whereas for some applications it would be interesting to consider particle systems with *local* slowdown. We believe that the techniques we develop in section 3 would carry over to such a setting with appropriate modifications; however, the difficulty of proving

convergence of such processes to the appropriate continuous object comes from proving uniqueness for the limiting SDE. We expect the solution of this SDE to spend a positive amount of time on lower-dimensional faces of the wedge $\mathcal{W}$, making the analysis of the process more difficult.

1.5. **Outline.** The rest of the paper is structured as follows. Section 2 is devoted to the study of sticky Brownian motions in $\mathcal{W}$. In subsection 2.1 we give the proof of existence and uniqueness of the weak solution to a system of SDEs generalizing (1.3). Then, in subsection 2.2 we show that the solution is a Markov process and study the invariant distributions of a suitably normalized version thereof. Subsequently, section 3 deals with the convergence of exclusion processes to sticky Brownian motions in $\mathcal{W}$. In section 3.1 we first prove Theorem 1, and then in section 3.2 we state and prove our main result, namely a generalized version of Theorem 1, which deals with the convergence of exclusion processes with non-exponential and possibly dependent jump interarrival times to sticky Brownian motions in $\mathcal{W}$.

2. Multidimensional sticky Brownian motions

This section is devoted to the study of the system of SDEs

$$\mathrm{d}X_i\left(t\right) = \mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\left(b_i\,\mathrm{d}t + \mathrm{d}W_i(t)\right) + \sum_{j=1}^{n-1} \mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}}\, v_{i,j}\,\mathrm{d}t, \tag{2.1}$$

$i \in [n]$, where b_i, $i \in [n]$ are real constants, $W = (W_1, W_2, \ldots, W_n)$ is an n-dimensional Brownian motion with zero drift vector and a strictly positive definite diffusion matrix $\mathfrak{C} = (\mathfrak{c}_{i,i'})_{i,i' \in [n]}$, and $V = (v_{i,j})_{i\in[n],\, j\in[n-1]}$ is a matrix with real entries. We note that the diffusion matrix of the process X is both discontinuous and degenerate, so neither existence nor uniqueness of a weak solution to (2.1) can be obtained directly from the classical results in [16] or [2].

2.1. **Existence and uniqueness.** In this subsection, we show that Assumption 1 is necessary and sufficient for the existence and uniqueness of a weak solution to (2.1). Furthermore, even under Assumption 1 one cannot expect a strong solution to exist.

Theorem 2. *Under Assumption* 1 *there exists a unique weak solution to* (2.1). *Moreover, if Assumption* 1 *does not hold, there is no weak solution to* (2.1).

Proof. Step 1. We start with the proof of weak existence. To this end, we first apply the main result in [17] to deduce that there exists a weak solution on a suitable filtered probability space $\left\{\Omega, \left(\mathcal{F}_t\right)_{t\geq 0}, P\right\}$ to the following system of SDEs:

$$\mathrm{d}\hat{Z}_i\left(t\right) = \left(b_{i+1} - b_i\right)\,\mathrm{d}t + \mathrm{d}B_i\left(t\right) + \sum_{j=1}^{n-1} q_{i,j}\,\mathrm{d}\Lambda_j\left(t\right), \quad i \in [n-1],$$

where $B = (B_1, B_2, \ldots, B_{n-1})$ is a Brownian motion with zero drift vector and diffusion matrix $A = (a_{i,i'})_{i,i'\in[n-1]}$ given by

$$a_{i,i'} = \mathfrak{c}_{i,i'} + \mathfrak{c}_{i+1,i'+1} - \mathfrak{c}_{i,i'+1} - \mathfrak{c}_{i+1,i'}, \tag{2.2}$$

and the $\Lambda_j\left(\cdot\right)$, $j \in [n-1]$ are the semimartingale local times at zero of the processes $\hat{Z}_j\left(\cdot\right)$, $j \in [n-1]$, respectively. Note that here we have used the fact that the

matrix Q is completely-$\mathcal{S}$ (see Assumption 1). Next, we can find (after extending the underlying probability space if necessary) a Brownian motion $\hat{\beta} = \big(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n\big)$ with zero drift vector and diffusion matrix $\mathfrak{C}$ such that

$$B_i(\cdot) = \hat{\beta}_{i+1}(\cdot) - \hat{\beta}_i(\cdot), \quad i \in [n-1].$$

Therefore, we can define $\hat{X} = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n)$ as the unique process satisfying

$$\sum_{i=1}^{n} \hat{X}_i(t) = \sum_{i=1}^{n} \Big(b_i\, t + \hat{\beta}_i(t) + \sum_{j=1}^{n-1} v_{i,j} \Lambda_j(t)\Big),$$
$$\big(\hat{X}_2\,(t) - \hat{X}_1\,(t)\,, \ldots, \hat{X}_n\,(t) - \hat{X}_{n-1}\,(t)\,\big) = \big(\hat{Z}_1\,(t)\,, \ldots, \hat{Z}_{n-1}\,(t)\,\big),$$

for all $t \geq 0$. Finally, we let

$$T(t) := t + \Lambda(t) := t + \sum_{j=1}^{n-1} \Lambda_j(t), \quad t \geq 0,$$
$$\tau(t) := \inf\{s \geq 0 :\ T(s) = t\}, \quad t \geq 0,$$

and set $X\,(\cdot) = \hat{X}\,(\tau\,(\cdot))$. Then clearly

$$X_i\,(\cdot) - X_i\,(0) = \int_0^{\tau(\cdot)} b_i\,\mathrm{d}t + \hat{\beta}_i\,(\tau(\cdot)) + \sum_{j=1}^{n-1} v_{i,j}\,\Lambda_j\,(\tau\,(\cdot))\,, \quad i \in [n].$$

Moreover, we note that $\tau\,(\cdot)$, $\Lambda\,(\tau\,(\cdot))$ are nondecreasing functions, which induce nonnegative measures $\mathrm{d}\tau\,(\cdot)$, $\mathrm{d}\Lambda\,(\tau\,(\cdot))$ on $[0,\infty)$ satisfying

$$\mathrm{d}\tau(t) + \mathrm{d}\Lambda(\tau(t)) = \mathrm{d}t. \tag{2.3}$$

Therefore, for $i \in [n]$ we have

$$\int_0^{\tau(\cdot)} b_i\,\mathrm{d}t = \int_0^{\tau(\cdot)} b_i\,\mathbf{1}_{\{\hat{X}_1(t)<\hat{X}_2(t)<\ldots<\hat{X}_n(t)\}}\,\mathrm{d}t = \int_0^{\cdot} b_i\,\mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\,\mathrm{d}\tau\,(t)$$
$$= \int_0^{\cdot} b_i\,\mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\,\mathrm{d}t.$$

In addition, the processes $\hat{\beta}_i(\tau(\cdot))$, $i \in [n]$ are martingales with respect to the filtration $(\mathcal{F}_{\tau(t)})_{t\geq 0}$ with quadratic covariation processes given by

$$\mathfrak{c}_{i,i'}\,\tau(\cdot) = \mathfrak{c}_{i,i'} \int_0^{\cdot} \mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\,\mathrm{d}t, \quad i, i' \in [n].$$

This identity can be derived by following the steps in the previous display. From the last computation we can conclude, in particular, that after extending the underlying probability space if necessary, we can find a Brownian motion $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ with zero drift vector and diffusion matrix $\mathfrak{C}$ such that

$$\hat{\beta}_i(\tau(\cdot)) = \int_0^{\cdot} \mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\,\mathrm{d}\beta_i\,(t)\,, \quad i \in [n].$$

Finally, we have

$$\begin{aligned}\Lambda_j\left(\tau\left(\cdot\right)\right) &= \int_0^{\cdot} \mathbf{1}_{\left\{\hat{X}_j(\tau(t))=\hat{X}_{j+1}(\tau(t))\right\}}\,\mathrm{d}\Lambda_j\left(\tau\left(t\right)\right) = \int_0^{\cdot} \mathbf{1}_{\left\{\hat{X}_j(\tau(t))=\hat{X}_{j+1}(\tau(t))\right\}}\left(\mathrm{d}t - \mathrm{d}\tau\left(t\right)\right)\\ &= \int_0^{\cdot} \mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}}\,\mathrm{d}t - \int_0^{\tau(\cdot)} \mathbf{1}_{\left\{\hat{X}_j(t)=\hat{X}_{j+1}(t)\right\}}\,\mathrm{d}t = \int_0^{\cdot} \mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}}\,\mathrm{d}t,\end{aligned}$$

for $j \in [n-1]$. Here the second identity is a consequence of (2.3) and the fact that the boundary local times $\Lambda_{j'}$, $j' \neq j$, do not charge the set $\{t:\ \hat{X}_j(t) = \hat{X}_{j+1}(t)\}$ (see the main result in [14]); and the fourth identity follows from the fact that the instantaneously reflecting Brownian motion $\hat{Z}$ does not spend time on the boundary of the orthant $\mathbb{R}_+^{n-1}$. All in all, we can now conclude that (X, β) is a weak solution to (2.1).

Step 2. We now turn to the proof of weak uniqueness. To this end, let (X, W) be any weak solution to (2.1). Define

$$\sigma(t) = \inf\left\{s \geq 0:\ \int_0^s \mathbf{1}_{\{X_1(a)<X_2(a)<\ldots<X_n(a)\}}\,\mathrm{d}a = t\right\}, \quad t \geq 0,$$

and set $\hat{X}\left(\cdot\right) = X\left(\sigma\left(\cdot\right)\right)$. Using Lévy's characterization of Brownian motion one verifies that

$$\hat{X}_i\left(t\right) = \hat{X}_i\left(0\right) + b_i\,t + \hat{W}_i\left(t\right) + \sum_{j=1}^{n-1} v_{i,j} L_j\left(t\right), \quad t \geq 0,$$

where $\hat{W} = (\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_n)$ is a Brownian motion with zero drift vector and diffusion matrix $\mathfrak{C}$, and $\{L_j\}_{j\in[n-1]}$ are nondecreasing processes whose points of increase are contained in the sets

$$\left\{t \geq 0:\ \hat{X}_j\left(t\right) = \hat{X}_{j+1}\left(t\right)\right\}, \quad j \in [n-1],$$

respectively. Moreover, the law of $\hat{X}$ is uniquely determined by the joint law of

$$\left(\hat{X}_2\left(\cdot\right) - \hat{X}_1\left(\cdot\right), \hat{X}_3\left(\cdot\right) - \hat{X}_2\left(\cdot\right), \ldots, \hat{X}_n\left(\cdot\right) - \hat{X}_{n-1}\left(\cdot\right)\right) \quad \text{and} \quad \sum_{i=1}^{n} \hat{X}_i(\cdot). \tag{2.4}$$

However, by the uniqueness result in [17] we can identify the first of the latter two processes as an instantaneously reflected Brownian motion in the orthant $\mathbb{R}_+^{n-1}$, so the joint law of that process and its boundary local times is uniquely determined. Moreover, the second process can be constructed by using the first process, its boundary local time processes and an additional independent one-dimensional standard Brownian motion, so the joint law of the processes in (2.4) is uniquely determined. Thus, the law of $\hat{X}$ is uniquely determined as well. Finally, the law of X is also uniquely determined as one can verify that $X\left(\cdot\right) = \hat{X}\left(\tau\left(\cdot\right)\right)$, where τ is defined as in step 1 above.

Step 3. Suppose now that Assumption 1 does not hold. Then a weak solution of (2.1) cannot exist. Indeed, if (X, W) was such a weak solution, we could define the time

change $\sigma(\cdot)$ as in step 2 above and let $\hat{X}(\cdot) = X(\sigma(\cdot))$ as before. Then, the arguments in step 2 would show that the process of spacings

$$\left(\hat{X}_2(\cdot) - \hat{X}_1(\cdot), \hat{X}_3(\cdot) - \hat{X}_2(\cdot), \ldots, \hat{X}_n(\cdot) - \hat{X}_{n-1}(\cdot)\right)$$

is a reflecting Brownian motion in the orthant $\mathbb{R}_+^{n-1}$ in the sense of [17]. However, by the main result in [17] the latter process does not exist if the reflection matrix Q is not completely-$\mathcal{S}$. This is the desired contradiction. □

The following example shows that, even when Assumption 1 holds, one cannot expect a strong solution to (2.1) to exist.

Example 1. Consider the following specification of parameters: $n = 2$, $b_1 = b_2 = 0$, $\mathfrak{c}_{1,1} = \mathfrak{c}_{2,2} = 1$, $\mathfrak{c}_{1,2} = \mathfrak{c}_{2,1} = 0$, $v_{1,1} = -\frac{1}{2}$, $v_{2,1} = \frac{1}{2}$; that is, the system of SDEs is

$$\mathrm{d}X_1(t) = \mathbf{1}_{\{X_1(t)<X_2(t)\}}\,\mathrm{d}W_1(t) - \frac{1}{2}\,\mathbf{1}_{\{X_1(t)=X_2(t)\}}\,\mathrm{d}t, \tag{2.5}$$

$$\mathrm{d}X_2(t) = \mathbf{1}_{\{X_1(t)<X_2(t)\}}\,\mathrm{d}W_2(t) + \frac{1}{2}\,\mathbf{1}_{\{X_1(t)=X_2(t)\}}\,\mathrm{d}t, \tag{2.6}$$

with W_1, W_2 being independent one-dimensional standard Brownian motions. We claim that this system does not admit a strong solution. It is well-known (see Theorem 3.2 in [4]) that strong existence and weak uniqueness together imply pathwise uniqueness, so it suffices to show that pathwise uniqueness does not hold for the system (2.5)-(2.6). To this end, we consider the SDE

$$\mathrm{d}Z(t) = \mathbf{1}_{\{Z(t)>0\}}\,\mathrm{d}\beta\,(t) + \mathbf{1}_{\{Z(t)=0\}}\,\mathrm{d}t, \tag{2.7}$$

where β is a Brownian motion with zero drift and diffusion coefficient 2. The main result in [5] shows that pathwise uniqueness does not hold for this equation. Therefore it suffices to argue that pathwise uniqueness for the system (2.5)-(2.6) would imply pathwise uniqueness for the equation (2.7). Indeed, let Z, Z' be two solutions of (2.7) on the same probability space and with respect to the same Brownian motion β. Extend the probability space so that it supports an independent Brownian motion W with zero drift and diffusion coefficient 2, and define S, S' according to

$$\mathrm{d}S\,(t) = \mathbf{1}_{\{Z(t)>0\}}\,\mathrm{d}W\,(t) \quad \text{and} \quad \mathrm{d}S'\,(t) = \mathbf{1}_{\{Z'(t)>0\}}\,\mathrm{d}W(t).$$

Finally, set

$$X_1 = \frac{S-Z}{2}, \quad X_2 = \frac{S+Z}{2} \quad \text{and} \quad X_1' = \frac{S'-Z'}{2}, \quad X_2' = \frac{S'+Z'}{2}.$$

Then both (X_1, X_2) and (X_1', X_2') are weak solutions of the system (2.5)-(2.6) with respect to the Brownian motion $((W-\beta)/2, (W+\beta)/2)$. Therefore if pathwise uniqueness did hold for the system (2.5)-(2.6), we would be able to conclude that $X_1 = X_1'$ and $X_2 = X_2'$ pathwise, and, hence, that $Z = Z'$ pathwise; in other words, the solution of (2.7) would be pathwise unique. This is the desired contradiction.

2.2. **Markov property and invariant measures.** Having established that the weak solution X of the system (2.1) exists and is unique (see Theorem 2), we can now proceed to study some of its properties. First, we remark that weak existence and uniqueness imply that the corresponding martingale problem is well-posed (see, e.g., Corollary 4.8 and Corollary 4.9 in chapter 5 of [12]). Therefore, by Theorem 6.2.2

in [16], the process X is Markovian. In addition, the relation $X(\cdot) = \hat{X}(\tau(\cdot))$, where $\hat{X}$ is an instantaneously reflecting Brownian motion in the wedge $\mathcal{W}$ with a nondegenerate diffusion matrix, shows that the process X has the Harris property:

$$\forall\, x,\, y \in \mathcal{W},\; r > 0: \quad \mathbb{P}^x\left(|X\left(t\right) - y| < r \text{ for some } t \geq 0\right) > 0. \tag{2.8}$$

Moreover, the corresponding property is true for the process of spacings

$$Z\left(\cdot\right) = \left(X_2\left(\cdot\right) - X_1\left(\cdot\right), X_3\left(\cdot\right) - X_2\left(\cdot\right), \ldots, X_n\left(\cdot\right) - X_{n-1}\left(\cdot\right)\right).$$

Thus, Z has a unique invariant distribution provided that it is recurrent or, equivalently, if

$$\hat{Z}\left(\cdot\right) = \left(\hat{X}_2\left(\cdot\right) - \hat{X}_1\left(\cdot\right), \hat{X}_3\left(\cdot\right) - \hat{X}_2\left(\cdot\right), \ldots, \hat{X}_n\left(\cdot\right) - \hat{X}_{n-1}\left(\cdot\right)\right)$$

is recurrent. By Proposition 2.8 in the dissertion [10], the latter is the case if and only if

$$Q^{-1}\left(b_2 - b_1, b_3 - b_2, \ldots, b_n - b_{n-1}\right)^T < 0 \tag{2.9}$$

componentwise. Here, the superscript T stands for the transpose of the vector under consideration. We summarize our findings in the next proposition.

Proposition 3. *The processes X, Z are Markovian. Both of them possess the Harris property. Moreover, the process Z has a unique invariant distribution if and only if the recurrence condition* (2.9) *is satisfied.*

For a wide class of coefficients the invariant distribution of the process Z can be given explicitly.

Theorem 4. *Suppose that in addition to* (2.9) *the condition*

$$2A = QD + DQ$$

is satisfied, where A is given by (2.2) *and $D = \operatorname{diag}(A)$ (the diagonal matrix, whose diagonal elements coincide with those of A). Then the invariant distribution of the process Z is given by*

$$\frac{1}{C}\Big(e^{-\langle\gamma,z\rangle}\,\mathrm{d}z + \sum_{j=1}^{n-1} e^{-\left\langle\gamma^j,z\right\rangle}\mathbf{1}_{\{z_j=0\}}\,\mathrm{d}z^j\Big),$$

where $C = \frac{1+\gamma_1+\ldots+\gamma_{n-1}}{\gamma_1\cdots\gamma_{n-1}}$ is the appropriate normalization constant,

$$\gamma = 2\operatorname{diag}\left(Q\right)D^{-1}Q^{-1}\left(b_2 - b_1, b_3 - b_2, \ldots, b_n - b_{n-1}\right)^T,$$
$$\gamma^j = \left(\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_{n-1}\right), \quad j \in \left[n-1\right],$$

and $\mathrm{d}z^j$, $j \in [n-1]$, are the Lebesgue boundary measures on the faces $\{z_j = 0\}$, $j \in [n-1]$, respectively.

Proof. Consider the transformed process $\tilde{Z}(\cdot) = A^{-1/2}\,Z(\cdot)$ which is a sticky Brownian motion in the cone $A^{-1/2}\left(\mathbb{R}_+\right)^{n-1}$ with drift vector $A^{-1/2}\left(b_2 - b_1, \ldots, b_n - b_{n-1}\right)^T$,

identity diffusion matrix and reflection matrix $A^{-1/2}Q$. Let $\mathcal{L}$ be the generator of the corresponding diffusion. Then, it is sufficient to check that

(2.10)
$$\forall f \in C_c^\infty\left(A^{-1/2}\left(\mathbb{R}_+\right)^{n-1}\right): \qquad \int\limits_{A^{-1/2}\left(\mathbb{R}_+\right)^{n-1}} \left(\mathcal{L}f\right) \mathrm{d}\nu + \sum_{j=1}^{n-1} \int\limits_{A^{-1/2}\{z_j=0\}} \left(\mathcal{L}f\right) \mathrm{d}\nu^j = 0,$$

where ν and ν^j, $j \in [n-1]$, are measures on $A^{-1/2}\left(\mathbb{R}_+\right)^{n-1}$ and $A^{-1/2}\{z_j = 0\}$, $j \in [n-1]$, respectively, induced by the measures $\frac{1}{C}e^{-\langle\gamma,z\rangle}\,\mathrm{d}z$ and $\frac{1}{C}e^{-\left\langle\gamma^j,z\right\rangle}\mathbf{1}_{\{z_j=0\}}\,\mathrm{d}z^j$, $j \in [n-1]$, respectively. From this point on, one just needs to follow the computations in the proof of Lemma 2.1 on page 130 in [9] to check that (2.10) is indeed satisfied. $\square$

3. Convergence and general setup

This section is divided into two parts. In the first part (subsection 3.1) we prove the convergence theorem (Theorem 1) as stated in the introduction. Then in the second part (subsection 3.2) we describe a much larger class of particle systems that converge to appropriate sticky Brownian motions in $\mathcal{W}$.

3.1. Proof of the convergence theorem.

Given the uniqueness of a weak solution to the system of SDEs (1.3) as proved in Theorem 2, Theorem 1 is a consequence of Proposition 5 below. To state and obtain the latter, we study the following decomposition. For each $i \in [n]$ we can write

(3.1)
$$X_i^M(t) = X_i^M(0) + A_i^M(t) + \sum_{j=1}^{n-1} C_{i,j}^{R,M}(t) - \sum_{j=1}^{n-1} C_{i,j}^{L,M}(t) + \sum_{j=1}^{n-1} \Delta_{i,j}^{R,M}(t) - \sum_{j=1}^{n-1} \Delta_{i,j}^{L,M}(t),$$

where

$$\begin{aligned}
A_i^M(t) &:= \int_0^t \mathbf{1}_{\left\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s),k=1,2,\ldots,n-1\right\}} \mathrm{d}\left(P_i(s) - Q_i(s)\right),\\
C_{i,j}^{R,M}(t) &:= \theta_{i,j}^R\, I_{i,j}^{R,M}(t) := \theta_{i,j}^R \int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \mathrm{d}s,\\
\Delta_{i,j}^{R,M}(t) &:= \int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \mathrm{d}\left(R_{i,j}(s) - \theta_{i,j}^R s\right),
\end{aligned}$$

and the processes $C_{i,j}^{L,M}$, $I_{i,j}^{L,M}$, and $\Delta_{i,j}^{L,M}$ are defined similarly to $C_{i,j}^{R,M}$, $I_{i,j}^{R,M}$, and $\Delta_{i,j}^{R,M}$, respectively. For $m \in \mathbb{N}$, let $D^m \equiv D\left([0,\infty),\mathbb{R}^m\right)$. We have the following convergence result.

Proposition 5. *Assume that the initial conditions $\left\{X^M(0), M > 0\right\}$ are deterministic and converge to a limit $x \in \mathcal{W}$ as $M \to \infty$. Then the family*

$$\left\{\left(X^M, A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M}\right), M > 0\right\} \tag{3.2}$$

is tight in D^{4n^2-2n}*. Moreover, every limit point* $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ *satisfies the following for each* $i \in [n]$ *and* $j \in [n-1]$*:*

$$X_i^\infty(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}} \sqrt{2a}\,\mathrm{d}W_i(s) + \sum_{j=1}^{n-1} v_{i,j} \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s, \tag{3.3}$$

$$A_i^\infty(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<X_2^\infty(s)<\cdots<X_n^\infty(s)\right\}} \sqrt{2a}\,\mathrm{d}W_i(s), \tag{3.4}$$

$$I_{i,j}^{L,\infty}(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s, \tag{3.5}$$

$$I_{i,j}^{R,\infty}(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s, \tag{3.6}$$

$$\Delta_{i,j}^{L,\infty}(\cdot) = \Delta_{i,j}^{R,\infty}(\cdot) = 0,$$

with a suitable n*-dimensional standard Brownian motion* $W = (W_1, \ldots W_n)$.

Proof. Step 1. The tightness of the family in (3.2) can be verified using the necessary and sufficient conditions of Corollary 3.7.4 in [7]. Indeed, note that for $i \in [n]$ and $j \in [n-1]$, the processes $P_i(\cdot) - Q_i(\cdot)$, as well as $\left(M^{1/4}(R_{i,j}(t) - \theta_{i,j}^R t), t \geq 0\right)$ and $\left(M^{1/4}(L_{i,j}(t) - \theta_{i,j}^L t), t \geq 0\right)$ all converge to suitable one-dimensional Brownian motions in the limit $M \to \infty$. Therefore, the conditions of Corollary 3.7.4 in [7] hold for the corresponding families of processes indexed by $M > 0$. It is now straightforward to verify that the same conditions hold for the family $(A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M})$, $M > 0$, so it is tight on D^{4n^2-3n}. In view of the decomposition (3.1), the first statement of the proposition now readily follows.

Step 2. Now, fix a limit point $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ and to simplify notation assume that it is the limit of the whole family (3.2) as $M \to \infty$.

We start with a few simple observations about the limit point under consideration. Note first that, for any fixed $M > 0$, the jumps of all components of $\left(X^M, A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M}\right)$ are bounded in absolute value by $\frac{1}{\sqrt{M}}$, so all components of the limit point must have continuous paths. Moreover, for every fixed $t \geq 0$, the family $\left\{A^M(t), M > 0\right\}$ is uniformly integrable due to the estimate

$$\mathbb{E}\left[A_i^M(t)^2\right] = \mathbb{E}\left[\int_0^t \mathbf{1}_{\left\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s),\,k\in[n-1]\right\}}\,\mathrm{d}\left[P_i - Q_i\right](s)\right] \leq 2\,a\,t, \quad i \in [n],$$

where $[\cdot]$ denotes the quadratic variation process of a process with paths in D^1. This and the fact that A^M is a martingale for any fixed $M > 0$ show that A^∞ is a martingale with respect to its own filtration.

Next, we observe that, as limits of non-decreasing processes, $I_{i,j}^{L,\infty}$ and $I_{i,j}^{R,\infty}$ must be non-decreasing processes themselves for every $i \in [n]$, $j \in [n-1]$. Finally, for all $i \in [n]$, $j \in [n-1]$, the quadratic variation processes of the martingales $\Delta_{i,j}^{L,M}$ and $\Delta_{i,j}^{R,M}$ satisfy

$$\forall\, t \geq 0: \quad \lim_{M\to\infty} \mathbb{E}\left[\left[\Delta_{i,j}^{L,M}\right](t)\right] = 0 \quad \text{and} \quad \lim_{M\to\infty} \mathbb{E}\left[\left[\Delta_{i,j}^{R,M}\right](t)\right] = 0.$$

Therefore, the distributional limits

$$\Delta_{i,j}^{L,\infty} \equiv \lim_{M\to\infty} \Delta_{i,j}^{L,M}, \quad \Delta_{i,j}^{R,\infty} \equiv \lim_{M\to\infty} \Delta_{i,j}^{R,M}$$

in D^1 exist and are identically equal to zero.

Step 3. To show (3.4), we study the quadratic covariation processes $\langle X_i^\infty, X_{i'}^\infty \rangle = \langle A_i^\infty, A_{i'}^\infty \rangle$, $i, i' \in [n]$. We first claim that $\langle A_i^\infty, A_{i'}^\infty \rangle = 0$ whenever $i \neq i'$. To this end, it suffices to show that for any such pair of indices $A_i^\infty(\cdot)A_{i'}^\infty(\cdot)$ is a martingale with respect to its own filtration. The latter is the limit in D^1 of the family of martingales $\left\{A_i^M(\cdot)A_{i'}^M(\cdot), M>0\right\}$ by definition, so it is enough to prove that, for any fixed $t \geq 0$, the random variables $\left\{A_i^M(t)A_{i'}^M(t), M>0\right\}$ are uniformly integrable. The latter is a consequence of the following chain of estimates:

$$\begin{aligned}
\mathbb{E}\left[A_i^M(t)^2 A_{i'}^M(t)^2\right] &= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \,\mathrm{d}A_{i'}^M(s)^2\right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \,\mathrm{d}A_i^M(s)^2\right] \\
&= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \,\mathrm{d}\left[A_{i'}^M\right](s)\right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \,\mathrm{d}\left[A_i^M\right](s)\right] \\
&\leq \mathbb{E}\left[\int_0^t A_i^M(s)^2 \,\mathrm{d}\left[P_{i'} - Q_{i'}\right](s)\right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \,\mathrm{d}\left[P_i - Q_i\right](s)\right] \\
&= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \frac{1}{\sqrt{M}}(\mathrm{d}P_{i'} + \mathrm{d}Q_{i'})(s)\right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \frac{1}{\sqrt{M}}(\mathrm{d}P_i + \mathrm{d}Q_i)(s)\right] \\
&= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \, 2a \,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \, 2a \,\mathrm{d}s\right] \\
&\leq 2a \int_0^t \mathbb{E}\left[\left[P_i - Q_i\right](s)\right] + \mathbb{E}\left[\left[P_{i'} - Q_{i'}\right](s)\right] \,\mathrm{d}s \leq 2a \int_0^t 4as \,\mathrm{d}s = 4a^2t^2.
\end{aligned}$$

We next aim to evaluate $\langle A_i^\infty \rangle$, $i \in [n]$. To this end, we note first that, for any fixed $i \in [n]$ and $t \geq 0$, the random variables $\left\{A_i^M(t)^2, M>0\right\}$ are uniformly integrable due to the estimate

$$\begin{aligned}
\mathbb{E}\left[A_i^M(t)^4\right] \leq\; & \mathbb{E}\left[\int_0^t 4\, A_i^M(s)^3 \,\mathrm{d}A_i^M(s)\right] \\
& + \mathbb{E}\left[\int_0^t \left(6\, A_i^M(s)^2 \frac{1}{\sqrt{M}} + 4|A_i^M(s)| \frac{1}{M} + \frac{1}{M^{3/2}}\right) \mathrm{d}(P_i + Q_i)(s)\right].
\end{aligned}$$

The first expectation on the right-hand side is equal to zero, since A_i^M is a martingale starting at zero; whereas the second expectation can be bounded above by a constant depending only on a and t by arguing as in the preceeding paragraph. Putting this together with the fact that the functional

$$(\omega_1, \omega_2, \ldots, \omega_n) \mapsto \int_{t_1}^{t_2} \mathbf{1}_{\{\omega_1(s) < \cdots < \omega_n(s)\}} \,\mathrm{d}s$$

on D^n is lower semicontinuous and using the Portmanteau Theorem, we have that

$$\begin{aligned}
&\mathbb{E}\left[G\left(A^{\infty}\right)\left(A_i^{\infty}\left(t_2\right)^2-A_i^{\infty}\left(t_1\right)^2-2a\int_{t_1}^{t_2}\mathbf{1}_{\{X_1^{\infty}(s)<\ldots<X_n^{\infty}(s)\}}\,\mathrm{d}s\right)\right]\\
&=\lim_{\epsilon\downarrow 0}\mathbb{E}\left[G\left(A^{\infty}\right)\left(A_i^{\infty}\left(t_2\right)^2-A_i^{\infty}\left(t_1\right)^2-2a\int_{t_1}^{t_2}\mathbf{1}_{\{X_k^{\infty}(s)+\epsilon<X_{k+1}^{\infty}(s),\,k\in[n-1]\}}\,\mathrm{d}s\right)\right]\\
&\geq\limsup_{M\to\infty}\mathbb{E}\left[G\left(A^{M}\right)\left(A_i^{M}\left(t_2\right)^2-A_i^{M}\left(t_1\right)^2-2a\int_{t_1}^{t_2}\mathbf{1}_{\{X_k^{M}(s)+\frac{1}{\sqrt{M}}<X_{k+1}^{M}(s),k\in[n-1]\}}\mathrm{d}s\right)\right]\\
&=\limsup_{M\to\infty}\mathbb{E}\left[G\left(A^{M}\right)\left(A_i^{M}\left(t_2\right)^2-A_i^{M}\left(t_1\right)^2-\int_{t_1}^{t_2}\mathbf{1}_{\{X_k^{M}(s)+\frac{1}{\sqrt{M}}<X_{k+1}^{M}(s),k\in[n-1]\}}\mathrm{d}[P_i-Q_i](s)\right)\right]\\
&\geq\limsup_{M\to\infty}\mathbb{E}\left[G\left(A^{M}\right)\left(A_i^{M}\left(t_2\right)^2-A_i^{M}\left(t_1\right)^2-\left[A_i^{M}\right]\left(t_2\right)+\left[A_i^{M}\right]\left(t_1\right)\right)\right]=0
\end{aligned}$$

for any nonnegative continuous bounded functional G on D^n measurable with respect to the σ-algebra generated by the coordinate mappings on $D^n\left([0,t_1]\right)$. Therefore, recalling that $\langle X_i^{\infty}\rangle=\langle A_i^{\infty}\rangle$, we conclude that

$$\forall\, 0\leq t_1<t_2:\quad \left\langle X_i^{\infty}\right\rangle\left(t_2\right)-\left\langle X_i^{\infty}\right\rangle\left(t_1\right)\geq 2a\int_{t_1}^{t_2}\mathbf{1}_{\left\{X_1^{\infty}(s)<\ldots<X_n^{\infty}(s)\right\}}\,\mathrm{d}s \tag{3.7}$$

holds with probability one. On the other hand,

$$\begin{aligned}
&\mathbb{E}\left[G\left(A^{\infty}\right)\left(A_i^{\infty}\left(t_2\right)^2-A_i^{\infty}\left(t_1\right)^2-2a(t_2-t_1)\right)\right]\\
&\qquad=\lim_{M\to\infty}\mathbb{E}\left[G\left(A^{M}\right)\left(A_i^{M}\left(t_2\right)^2-A_i^{M}\left(t_1\right)^2-2a(t_2-t_1)\right)\right]\\
&\qquad=\lim_{M\to\infty}\mathbb{E}\left[G\left(A^{M}\right)\left(A_i^{M}\left(t_2\right)^2-A_i^{M}\left(t_1\right)^2-[P_i-Q_i](t_2)+[P_i-Q_i](t_1)\right)\right]\leq 0
\end{aligned}$$

for any functional G on D^n as above. Hence,

$$\forall\, 0\leq t_1<t_2:\quad \left\langle X_i^{\infty}\right\rangle\left(t_2\right)-\left\langle X_i^{\infty}\right\rangle\left(t_1\right)\leq 2a\left(t_2-t_1\right) \tag{3.8}$$

must hold with probability 1.

In view of (3.8), we see that in order to improve (3.7) to an equality, it suffices to show that the measure $\mathrm{d}\left\langle X_i^{\infty}\right\rangle=\mathrm{d}\left\langle A_i^{\infty}\right\rangle$ assigns zero mass to the sets $\{t\geq 0: X_j^{\infty}(t)=X_{j+1}^{\infty}(t)\}$, $j\in[n-1]$, with probability one. To this end, we first recall that for every $i\in[n]$ the square integrable martingale A_i^{∞} is the limit in D^1 of the square integrable martingales $\left\{A_i^M, M>0\right\}$, and the random variables $\left\{A_i^M(t)^2, M>0\right\}$ are uniformly integrable for any fixed $t\geq 0$. Therefore $\langle A_i^{\infty}\rangle$ is the limit in D^1 of $\left\{\left[A_i^M\right], M>0\right\}$, and so by the Portmanteau Theorem

$$\begin{aligned}
&\mathbb{E}\left[\left(\left\langle A_i^{\infty}\right\rangle(t)-\left\langle A_{i'}^{\infty}\right\rangle(t)\right)^2\right]\leq\liminf_{M\to\infty}\mathbb{E}\left[\left(\left[A_i^{M}\right](t)-\left[A_{i'}^{M}\right](t)\right)^2\right]\\
&\leq\liminf_{M\to\infty}\mathbb{E}\Big[\Big(\int_0^t\mathbf{1}_{\{X_k^{M}(s)+\frac{1}{\sqrt{M}}<X_{k+1}^{M}(s),k\in[n-1]\}}\frac{1}{\sqrt{M}}\left(\mathrm{d}P_i+\mathrm{d}Q_i-\mathrm{d}P_{i'}-\mathrm{d}Q_{i'}\right)(s)\Big)^2\Big]\\
&=\liminf_{M\to\infty}\mathbb{E}\left[\int_0^t\mathbf{1}_{\{X_k^{M}(s)+\frac{1}{\sqrt{M}}<X_{k+1}^{M}(s),k\in[n-1]\}}\frac{1}{M}\,\mathrm{d}\left[P_i+Q_i-P_{i'}-Q_{i'}\right](s)\right]\\
&\leq\liminf_{M\to\infty}\frac{1}{M^{3/2}}\,\mathbb{E}[P_i(t)+Q_i(t)+P_{i'}(t)+Q_{i'}(t)]=0
\end{aligned}$$

for any fixed $i, i' \in [n]$ and $t \geq 0$ with probability one. In view of the path continuity of the processes $\langle X_i^\infty \rangle$, $i \in [n]$, this implies

$$\langle X_1^\infty \rangle = \langle X_2^\infty \rangle = \ldots = \langle X_n^\infty \rangle$$

with probability one. To conclude the argument, note that the occupation time formula for continuous semimartingales (see, e.g., [15, Theorem VI.1.6]) implies that the measure

$$\mathrm{d}\left\langle X_{j+1}^\infty - X_j^\infty \right\rangle = \mathrm{d}\left\langle X_{j+1}^\infty \right\rangle + \mathrm{d}\left\langle X_j^\infty \right\rangle = 2\,\mathrm{d}\left\langle X_j^\infty \right\rangle = 2\,\mathrm{d}\left\langle X_i^\infty \right\rangle$$

assigns zero mass to the set $\{t \geq 0 : X_j^\infty(t) = X_{j+1}^\infty(t)\}$ with probability one. Hence, equality must hold in (3.7). The representation (3.4) with a suitable standard Brownian motion $W = (W_1, W_2, \ldots, W_n)$ now readily follows from the Martingale Representation Theorem in the form of Theorem 4.2 in chapter 3 of [12].

Step 4. We now turn to the proof of (3.3), (3.5) and (3.6). To this end, for any $M > 0$, $i \in [n]$, $j \in [n-1]$, we consider the decompositions

$$\begin{aligned} I_{i,j}^{L,M}(t) &= I_j^{M,1}(t) - I_{i,j}^{L,M,2}(t) \\ &:= \int_0^t \mathbf{1}_{\left\{X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}}\,\mathrm{d}s - \int_0^t \mathbf{1}_{\left\{X_{i-1}^M(s)+\frac{1}{\sqrt{M}}=X_i^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}}\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} I_{i,j}^{R,M}(t) &= I_j^{M,1}(t) - I_{i,j}^{R,M,2}(t) \\ &:= \int_0^t \mathbf{1}_{\left\{X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}}\,\mathrm{d}s - \int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}=X_{i+1}^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}}\,\mathrm{d}s. \end{aligned}$$

It is now easy to check that for each $i \in [n]$, $j \in [n-1]$, the families of processes $\left\{I_j^{M,1}, M > 0\right\}$, $\left\{I_{i,j}^{L,M,2}, M > 0\right\}$ and $\left\{I_{i,j}^{R,M,2}, M > 0\right\}$ satisfy the tightness criterion of Corollary 3.7.4 in [7]. So, after passing to a subsequence if necessary, we obtain the existence of suitable limits $I_j^{\infty,1}$, $I_{i,j}^{L,\infty,2}$ and $I_{i,j}^{R,\infty,2}$, respectively; for notational convenience we assume that the full families of processes converge jointly to the respective limit points.

Clearly the limits are non-decreasing processes and inherit the properties

$$\forall\, 0 \leq t_1 < t_2 : \quad I_{i,j}^{L,\infty,2}(t_2) - I_{i,j}^{L,\infty,2}(t_1) \leq I_j^{\infty,1}(t_2) - I_j^{\infty,1}(t_1), \tag{3.9}$$

$$\forall\, 0 \leq t_1 < t_2 : \quad I_{i,j}^{R,\infty,2}(t_2) - I_{i,j}^{R,\infty,2}(t_1) \leq I_j^{\infty,1}(t_2) - I_j^{\infty,1}(t_1). \tag{3.10}$$

Moreover, arguing as in the second half of the proof of Theorem 4.1 in [20] (see also the proof of Proposition 9 in [11]), one obtains

$$\int_0^\infty \mathbf{1}_{\{X_j^\infty(t) < X_{j+1}^\infty(t)\}}\,\mathrm{d}I_j^{\infty,1}(t) = 0, \tag{3.11}$$

$$\int_0^\infty \left(\mathbf{1}_{\{X_{i-1}^\infty(t) < X_i^\infty(t)\}} + \mathbf{1}_{\{X_j^\infty(t) < X_{j+1}^\infty(t)\}}\right)\,\mathrm{d}I_{i,j}^{L,\infty,2}(t) = 0, \tag{3.12}$$

$$\int_0^\infty \left(\mathbf{1}_{\{X_i^\infty(t) < X_{i+1}^\infty(t)\}} + \mathbf{1}_{\{X_j^\infty(t) < X_{j+1}^\infty(t)\}}\right)\,\mathrm{d}I_{i,j}^{R,\infty,2}(t) = 0 \tag{3.13}$$

from the corresponding properties of the prelimit processes.

Next, we define the time change

$$\sigma(t) = \inf\left\{s \geq 0 : \int_0^s \mathbf{1}_{\{X_1^\infty(r) < X_2^\infty(r) < \ldots < X_n^\infty(r)\}}\,\mathrm{d}r = t\right\}, \quad t \geq 0$$

and let $\hat{X}^\infty(\cdot) = X^\infty(\sigma(\cdot))$. Using Lévy's characterization of Brownian motion, we conclude that the components of $\hat{X}^\infty$ admit the decomposition

$$\hat{X}_i^\infty(\cdot) = \hat{X}_i^\infty(0) + \hat{W}_i(\cdot) + \sum_{j=1}^{n-1} v_{i,j}\, I_j^{\infty,1}(\sigma(\cdot)) + \sum_{j=1}^{n-1}\Big(\theta_{i,j}^R\, I_{i,j}^{R,\infty,2}(\sigma(\cdot)) - \theta_{i,j}^L\, I_{i,j}^{L,\infty,2}(\sigma(\cdot))\Big)$$

with $\hat{W} = (\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_n)$ being a suitable standard Brownian motion. Moreover, arguing as in the proof of Lemma 1 in [11] (in particular, using the Lyapunov functions constructed in the proof of Lemma 4 in [14] as well as (3.9)-(3.13)), we can conclude that

$$I_{i,j}^{L,\infty,2}(\sigma(\cdot)) \equiv 0 \quad \text{and} \quad I_{i,j}^{R,\infty,2}(\sigma(\cdot)) \equiv 0.$$

Furthermore, we can identify the process of spacings

$$\Big(\hat{X}_2^\infty(\cdot) - \hat{X}_1^\infty(\cdot),\, \hat{X}_3^\infty(\cdot) - \hat{X}_2^\infty(\cdot),\, \ldots,\, \hat{X}_n^\infty(\cdot) - \hat{X}_{n-1}^\infty(\cdot)\Big)$$

as a reflected Brownian motion in the orthant $(\mathbb{R}_+)^{n-1}$ with reflection matrix Q (defined in Assumption 1), and the processes $I_j^{\infty,1}(\sigma(\cdot))$, $j \in [n-1]$, with its boundary local times. At this point, to obtain the representations (3.3), (3.5) and (3.6), one can argue as in step 2 in the proof of Theorem 2. □

3.2. **General setup.** In this last subsection, we introduce a much more general class of particle systems which converge to appropriate multidimensional sticky Brownian motions in the sense of Theorem 1. We now allow for non-exponential interarrival times between the jumps of the particles and for dependence between the arrival times of the jumps for different particles.

To define this more general class of particle systems, we introduce the following parameters: $n \in \mathbb{N}$ for the number of particles as before; $a > 0$; λ_i^L and λ_i^R for $i \in [n]$; $c_{i,i'}^{L,L}$, $c_{i,i'}^{L,R}$, and $c_{i,i'}^{R,R}$ for $i, i' \in [n]$; and finally $\theta_{i,j}^L$ and $\theta_{i,j}^R$ for $i \in [n]$, $j \in [n-1]$. We fix a value $M > 0$ of the scaling parameter. The random variables and processes we define next all depend on M, but for the sake of readability we mostly do not denote this dependence explicitly.

We let $\{u^L(k), k \in \mathbb{N}\}$ and $\{u^R(k), k \in \mathbb{N}\}$ be two independent sequences of i.i.d. random vectors with values in $(0,\infty)^n$ (the interarrival times between jumps to the left and to the right when there are no collisions), and for $i \in [n]$, $j \in [n-1]$, let $\{w_{i,j}^L(k), k \in \mathbb{N}\}$ and $\{w_{i,j}^R(k), k \in \mathbb{N}\}$ be two independent families of i.i.d. random variables taking values in $(0,\infty)$ (the interarrival times between jumps to the left and to the right when there is a collision). We assume that

$$\mathbb{E}[u_i^L(1)] = \left(a + \frac{\lambda_i^L}{\sqrt{M}}\right)^{-1}, \quad \mathbb{E}[u_i^R(1)] = \left(a + \frac{\lambda_i^R}{\sqrt{M}}\right)^{-1},$$
$$\mathrm{cov}(u_i^L(1), u_{i'}^L(1)) = c_{i,i'}^{L,L},\ \mathrm{cov}(u_i^L(1), u_{i'}^R(1)) = c_{i,i'}^{L,R},\ \mathrm{cov}(u_i^R(1), u_{i'}^R(1)) = c_{i,i'}^{R,R},$$
$$\mathbb{E}[w_{i,j}^L(1)] = \left(\theta_{i,j}^L\right)^{-1}, \quad \mathbb{E}[w_{i,j}^R(1)] = \left(\theta_{i,j}^R\right)^{-1}.$$

Next, define the corresponding partial sum processes

$$U_i^L(0)=0,\quad U_i^L(\ell)=\sum_{k=1}^{\ell}u_i^L(k),\quad U_i^R(0)=0,\quad U_i^R(\ell)=\sum_{k=1}^{\ell}u_i^R(k),$$
$$W_{i,j}^L(0)=0,\quad W_{i,j}^L(\ell)=\sum_{k=1}^{\ell}w_{i,j}^L(k),\quad W_{i,j}^R(0)=0,\quad W_{i,j}^R(\ell)=\sum_{k=1}^{\ell}w_{i,j}^R(k)$$

for all $i\in[n]$, $j\in[n-1]$, and also the corresponding renewal processes

$$S_i^L(t)=\max\left\{k\geq 0:\ U_i^L(k)\leq t\right\},\quad S_i^R(t)=\max\left\{k\geq 0:\ U_i^R(k)\leq t\right\},$$
$$T_{i,j}^L(t)=\max\left\{k\geq 0:\ W_{i,j}^L(k)\leq t\right\},\quad T_{i,j}^R(t)=\max\left\{k\geq 0:\ W_{i,j}^R(k)\leq t\right\}.$$

We now define the particle system for any fixed value of the scaling parameter $M>0$ according to

$$\begin{aligned}\mathrm{d}X_i^M(t)=&\ \frac{1}{\sqrt{M}}\,\mathbf{1}_{\left\{X_k^M(t)+\frac{1}{\sqrt{M}}<X_{k+1}^M(t),\,k\in[n-1]\right\}}\,\mathrm{d}\left(S_i^R(M\,t)-S_i^L(M\,t)\right)\\ &+\frac{1}{\sqrt{M}}\sum_{j=1}^{n-1}\mathbf{1}_{\left\{X_i^M(t)+\frac{1}{\sqrt{M}}<X_{i+1}^M(t),X_j^M(t)+\frac{1}{\sqrt{M}}=X_{j+1}^M(t)\right\}}\,\mathrm{d}T_{i,j}^R\left(\sqrt{M}\,t\right)\\ &-\frac{1}{\sqrt{M}}\sum_{j=1}^{n-1}\mathbf{1}_{\left\{X_{i-1}^M(t)+\frac{1}{\sqrt{M}}<X_i^M(t),X_j^M(t)+\frac{1}{\sqrt{M}}=X_{j+1}^M(t)\right\}}\,\mathrm{d}T_{i,j}^L\left(\sqrt{M}\,t\right),\end{aligned}\tag{3.14}$$

for $i\in[n]$. Note that the particle configuration $\left(X_1^M(t),X_2^M(t),\ldots,X_n^M(t)\right)$ is an element of the discrete wedge $\mathcal{W}^M$ for any $t\geq 0$. We also remark at this point that (3.14) generalizes (1.2). Indeed, if $\lambda_i^L=\lambda_i^R=0$ for $i\in[n]$, $c_{i,i'}^{L,L}=c_{i,i'}^{L,R}=c_{i,i'}^{R,R}=0$ whenever $i\neq i'$, $c_{i,i}^{L,L}=c_{i,i}^{R,R}=a^{-1}$ and $c_{i,i}^{L,R}=0$ for $i\in[n]$, and all interarrival times above are independent exponential random variables with appropriate means, then (3.14) reduces to (1.2).

For the extension of our convergence theorem to particle systems as in (3.14), we need the following moment assumption on the interarrival times between jumps.

Assumption 2. Assume that there exists $\delta>0$ such that

$$\sup_{M>0}\max_{i\in[n]}\left(\mathbb{E}\left[u_i^L(1)^{2+\delta}\right]+\mathbb{E}\left[u_i^R(1)^{2+\delta}\right]\right)<\infty,$$
$$\sup_{M>0}\max_{i\in[n],\,j\in[n-1]}\left(\mathbb{E}\left[w_{i,j}^L(1)^{2+\delta}\right]+\mathbb{E}\left[w_{i,j}^R(1)^{2+\delta}\right]\right)<\infty.$$

Under Assumption 2 we have the following convergence result, which generalizes Theorem 1.

Theorem 6. *Suppose that Assumptions* 1 *and* 2 *hold, and that the initial conditions* $\left\{X^M(0),M>0\right\}$ *are deterministic and converge to a limit* $x\in\mathcal{W}$ *as* $M\to\infty$. *Then the laws of the paths of the particle systems* $\left\{X^M(\cdot),M>0\right\}$ *defined in* (3.14) *converge in* $D([0,\infty),\mathbb{R}^n)$ *to the law of the unique weak solution of the system of*

SDEs
(3.15)

$$\mathrm{d}X_i\left(t\right) = \mathbf{1}_{\{X_1(t)<\ldots<X_n(t)\}}\,\left(\left(\lambda_i^R-\lambda_i^L\right)\,\mathrm{d}t + a^{3/2}\,\mathrm{d}W_i\left(t\right)\right) + \sum_{j=1}^{n-1}\mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}}\,v_{i,j}\,\mathrm{d}t$$

for $i\in[n]$, taking values in $\mathcal{W}$ and starting from x. Here, $W=(W_1,W_2,\ldots,W_n)$ is Brownian motion in $\mathbb{R}^n$ with zero drift vector and diffusion matrix given by

$$\mathfrak{C}=(\mathfrak{c}_{i,i'})=(c^{L,L}_{i,i'}+c^{L,R}_{i,i'}+c^{L,R}_{i',i}+c^{R,R}_{i',i'});$$

and $v_{i,j}=\theta^R_{i,j}-\theta^L_{i,j}$ is as in Assumption 1.

The existence and uniqueness of a weak solution to the system of SDEs (3.15) is proven in Theorem 2, so Theorem 6 is a consequence of Proposition 7 below, which is the appropriate generalization of Proposition 5 in subsection 3.1.

As in subsection 3.1, we need to study an appropriate decomposition of the particle dynamics. For each $i\in[n]$, we write

$$X_i^M\left(t\right)=X_i^M\left(0\right)+A_i^M\left(t\right)+\sum_{j=1}^{n-1}C_{i,j}^{R,M}\left(t\right)-\sum_{j=1}^{n-1}C_{i,j}^{L,M}\left(t\right)+\sum_{j=1}^{n-1}\Delta_{i,j}^{R,M}\left(t\right)-\sum_{j=1}^{n-1}\Delta_{i,j}^{L,M}\left(t\right),$$

where

$$\begin{aligned}
A_i^M\left(t\right) &:= \frac{1}{\sqrt{M}}\int_0^t \mathbf{1}_{\left\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s),\,k\in[n-1]\right\}}\,\mathrm{d}\left(S_i^R\left(M\,s\right)-S_i^L\left(M\,s\right)\right),\\
C_{i,j}^{R,M}\left(t\right) &:= \theta_{i,j}^R\,I_{i,j}^{R,M} := \theta_{i,j}^R\int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}}\,\mathrm{d}s\\
\Delta_{i,j}^{R,M}\left(t\right) &:= \frac{1}{\sqrt{M}}\int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}}\,\mathrm{d}\overline{T}_{i,j}^R\left(\sqrt{M}\,s\right),\\
\overline{T}_{i,j}^R\left(t\right) &:= T_{i,j}^R\left(t\right)-\theta_{i,j}^R\,t,
\end{aligned}$$

and the processes $C_{i,j}^{L,M}$, $I_{i,j}^{L,M}$, $\Delta_{i,j}^{L,M}$, and $\overline{T}_{i,j}^L$ are defined similarly to $C_{i,j}^{R,M}$, $I_{i,j}^{R,M}$, $\Delta_{i,j}^{R,M}$, and $\overline{T}_{i,j}^R$, respectively. The following proposition is the appropriate generalization of Proposition 5 to the present framework.

Proposition 7. *Suppose that Assumptions* 1 *and* 2 *hold, and that the initial conditions $\left\{X^M\left(0\right),M>0\right\}$ are deterministic and converge to a limit $x\in\mathcal{W}$ as $M\to\infty$. Then the family*

$$\left\{\left(X^M,A^M,I^{L,M},I^{R,M},\Delta^{L,M},\Delta^{R,M}\right),\ M>0\right\} \tag{3.16}$$

is tight in D^{4n^2-2n}*. Moreover, every limit point* $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ *satisfies the following for each* $i \in [n]$ *and* $j \in [n-1]$*:*

$$\begin{aligned} X_i^\infty(\cdot) = &\int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}} \left(\left(\lambda_i^R - \lambda_i^L\right) \mathrm{d}s + a^{3/2}\,\mathrm{d}W_i(s)\right) \\ &+ \sum_{j=1}^{n-1} v_{i,j} \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s, \end{aligned} \tag{3.17}$$

$$A_i^\infty(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}} \left(\left(\lambda_i^R - \lambda_i^L\right) \mathrm{d}s + a^{3/2}\,\mathrm{d}W_i(s)\right), \tag{3.18}$$

$$I_{i,j}^{L,\infty}(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s, \tag{3.19}$$

$$I_{i,j}^{R,\infty}(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s,, \tag{3.20}$$

$$\Delta_{i,j}^{L,\infty}(\cdot) = \Delta_{i,j}^{R,\infty}(\cdot) = 0,$$

with a Brownian motion $W = (W_1, W_2, \ldots, W_n)$ *as in the statement of Theorem* 6*.*

Proof. One can proceed as in the proof of Proposition 5, so we only explain the arguments which are different. First, note that Theorem 14.6 in [3] and its proof extend to the case of the multidimensional renewal processes

$$\left\{S_i^L\right\}_{i\in[n]}, \quad \left\{S_i^R\right\}_{i\in[n]}, \quad \left\{T_{i,j}^L\right\}_{i\in[n],j\in[n-1]}, \quad \left\{T_{i,j}^R\right\}_{i\in[n],j\in[n-1]},$$

yielding the joint convergence of

$$\begin{aligned} &\left\{\left(M^{-1/2}(S_i^R(M\,t) - S_i^L(M\,t)), t \ge 0\right)\right\}_{i\in[n]}, \\ &\left\{\left(M^{-1/4}\,\overline{T}_{i,j}^L(\sqrt{M}\,t), t \ge 0\right)\right\}_{i\in[n],j\in[n-1]}, \\ \text{and} \quad &\left\{\left(M^{-1/4}\,\overline{T}_{i,j}^R(\sqrt{M}\,t), t \ge 0\right)\right\}_{i\in[n],j\in[n-1]} \end{aligned}$$

to appropriate Brownian motions. The rest of step 1 and step 2 in the proof of Proposition 5 carry over to the present setting in a straightforward manner.

Now, one needs to show that every limit point $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ satisfies

$$\langle X_i^\infty, X_{i'}^\infty\rangle(\cdot) = \langle A_i^\infty, A_{i'}^\infty\rangle(\cdot) = a^3\,\mathfrak{c}_{\mathrm{i,i'}} \int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}}\,\mathrm{d}s.$$

To this end, one can first proceed as in step 3 in the proof of Proposition 5 to show that

$$\mathrm{d}\,\langle X_i^\infty, X_{i'}^\infty\rangle = \frac{\mathfrak{c}_{i,i'}}{\mathfrak{c}_{j,j}}\,\mathrm{d}\left\langle X_j^\infty\right\rangle, \quad i, i' \in [n],\ j \in [n]. \tag{3.21}$$

Next, one can invoke the Portmanteau Theorem as before to conclude that for all $i \in [n]$ and $0 \le t_1 < t_2$:

$$a^3\,\mathfrak{c}_{\mathrm{i,i}} \int_{t_1}^{t_2} \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}}\,\mathrm{d}s \le \langle X_i^\infty\rangle(t_2) - \langle X_i^\infty\rangle(t_1) \le a^3\,\mathfrak{c}_{\mathrm{i,i}}\,(t_2 - t_1). \tag{3.22}$$

Moreover, since the measures $\mathrm{d}\left\langle X_j^\infty\right\rangle$, $j \in [n-1]$, assign zero mass to the sets $\{t \geq 0 :\ X_j^\infty(t) = X_{j+1}^\infty(t)\}$, $j \in [n-1]$, respectively, (3.21) and (3.22) suffice to identify all quadratic covariation processes $\langle X_i^\infty, X_{i'}^\infty\rangle$, $i, i' \in [n]$. Similarly, one can identify the bounded variation parts of the processes A_i^∞, $i \in [n]$, as multiples of the quadratic variation processes $\langle X_i^\infty\rangle$, $i \in [n]$, respectively. The rest of the proof can be carried out by following the arguments in step 4 in the proof of Proposition 5. $\square$

Acknowledgement

The authors thank Soumik Pal for many helpful discussions in the course of preparation of this paper.

References

[1] M. Amir. Sticky Brownian motion as the strong limit of a sequence of random walks. *Stochastic Processes and their Applications*, 39(2):221–237, 1991.

[2] R.F. Bass and É. Pardoux. Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields*, 76(4):557–572, 1987.

[3] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999.

[4] A.S. Cherny. On the uniqueness in law and the pathwise uniqueness for stochastic differential equations. *Theory of Probability & Its Applications*, 46(3):406–419, 2002.

[5] R.J. Chitashvili. On the nonexistence of a strong solution in the boundary problem for a sticky Brownian motion. *Centrum voor Wiskunde en Informatica, Centre for Mathematics and Computer Science, Report BS-R8901*, 1989.

[6] M. Émery and W. Schachermayer. On Vershik's standardness criterion and Tsirelson's notion of cosiness. *Séminaire de Probabilités XXXV, Springer Lecture Notes in Mathematics*, 1755:265–305, 2001.

[7] S. N. Ethier and T. G. Kurtz. *Markov Processes: Characterization and Convergence*. Wiley & Sons, 1986.

[8] J.M. Harrison and A.J. Lemoine. Sticky Brownian motion as the limit of storage processes. *Journal of Applied Probability*, 18:216–226, 1981.

[9] J.M. Harrison and R.J. Williams. Multidimensional reflected Brownian motions having exponential stationary distributions. *Ann. Probab.*, 15(1):115–137, 1987.

[10] T. Ichiba. *Topics in multidimensional diffusion theory: attainability, reflection, ergodicity and rankings*. Dissertation, Columbia University. 2009.

[11] I. Karatzas, S. Pal, and M. Shkolnikov. Systems of Brownian particles with asymmetric collisions. *Arxiv preprint arXiv:1210.0259*, 2012.

[12] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[13] A.J. Lemoine. Limit theorems for generalized single server queues: the exceptional system. *SIAM J. Appl. Math.*, 28(3):596–606, 1975.

[14] M.I. Reiman and R.J. Williams. A boundary property of semimartingale reflecting Brownian motions. *Probab. Theory Related Fields*, 77(1):87–97, 1988.

[15] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[16] D.W. Stroock and S.R.S. Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[17] L.M. Taylor and R.J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields*, 96(3):283–317, 1993.

[18] J. Warren. Branching processes, the Ray-Knight theorem, and sticky Brownian motion. *Séminaire de Probabilités XXXI*, pages 1–15, 1997.
[19] R.J. Williams. Semimartingale reflecting Brownian motions in the orthant. In *Stochastic networks*, volume 71 of *IMA Vol. Math. Appl.*, pages 125–137. Springer, New York, 1995.
[20] R.J. Williams. An invariance principle for semimartingale reflecting Brownian motions in an orthant. *Queueing Systems*, 30:5–25, 1998.

University of California, Berkeley, Department of Statistics, Berkeley, CA 94720
E-mail address: `racz@stat.berkeley.edu`

University of California, Berkeley, Department of Statistics, Berkeley, CA 94720
E-mail address: `mshkolni@gmail.com`